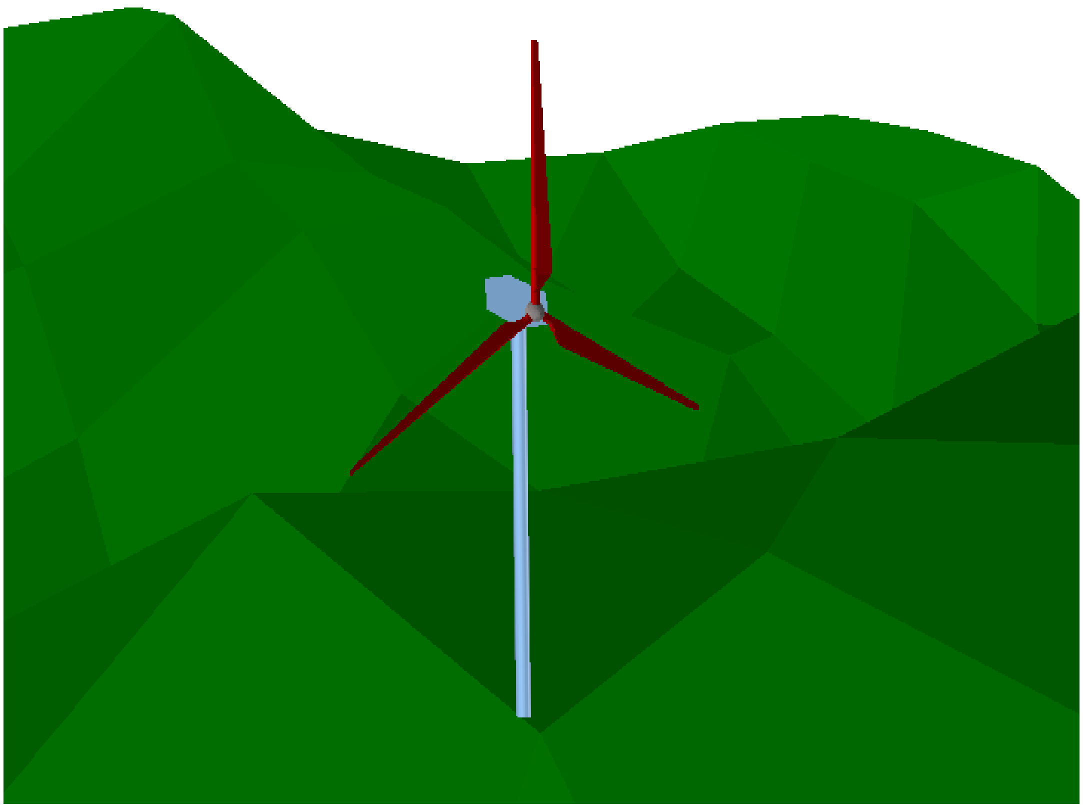
Response to Harmonic Excitation

Part 2: Damped Systems



Part 1 covered the response of a single degree of freedom system to harmonic excitation without considering the effects of damping. However, almost every real world system that is analyzed from a vibrations point of view has some source of energy dissipation or damping. This module will expand on Part 1 and discuss the response of such a system to harmonic excitations with the effects of viscous damping included.

Harmonic Excitation of Damped Systems

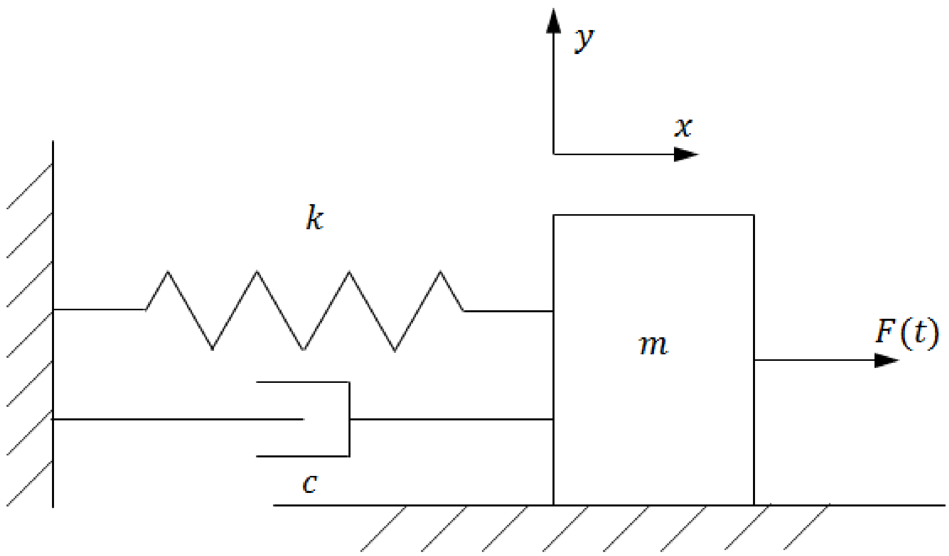


Fig 1: Spring-mass-damper with external force

Consider a simple spring-mass system with damping being driven by a force $F(t)$ of the form $F(t) = F_0 \cdot \cos(\omega \cdot t)$ on a frictionless surface. Here F_0 is the magnitude of the applied force and ω is the angular frequency of the applied force. The sum of the forces in the y-direction is 0, resulting in no motion in that direction. In the x-direction, summing the forces on the mass yields

$$m \cdot \ddot{x}(t) + c \cdot \dot{x}(t) + k \cdot x(t) = F_0 \cdot \cos(\omega \cdot t)$$

... Eq. (1)

Dividing through by m gives

$$\ddot{x}(t) + 2 \cdot \zeta \cdot \omega_n \cdot \dot{x}(t) + \omega_n^2 \cdot x(t) = f_0 \cdot \cos(\omega \cdot t)$$

... Eq. (2)

where, as a reminder,

$f_0 = \frac{F_0}{m}$, $\omega_n = \sqrt{\frac{k}{m}}$ is the natural frequency of the system and

$\zeta = \frac{c}{2 \cdot \sqrt{k \cdot m}}$ is the damping ratio. There are many ways of finding a solution to this type of a differential equation (Laplace transform approach, geometric method, method of undetermined coefficients, etc.). Here, the method of undetermined coefficients will be used by assuming the particular solution has the form shown in Eq. (3). It is known, from observation and from the study of differential equations, that the forced response of a damped system has the same frequency as the driving force but with a different amplitude and phase. Therefore, the particular solution of Eq. (2) is assumed to be of the form

$$x_p(t) = X \cdot \cos(\omega \cdot t - \theta)$$

... Eq. (3)

To simplify the computations, this can be written as

$$x_p(t) = A_s \cdot \cos(\omega \cdot t) + B_s \cdot \sin(\omega \cdot t)$$

... Eq. (4)

where $A_s = X \cdot \cos(\theta)$ and $B_s = X \cdot \sin(\theta)$ such that $X = \sqrt{A_s + B_s}$ and $\theta = \tan^{-1}\left(\frac{B_s}{A_s}\right)$. Taking derivatives of Eq. (4) we get

$$\dot{x}_p(t) = -\omega \cdot A_s \cdot \sin(\omega \cdot t) + \omega \cdot B_s \cdot \cos(\omega \cdot t)$$

... Eq. (5)

and

$$\ddot{x}_p(t) = -\omega^2 \cdot A_s \cdot \cos(\omega \cdot t) - \omega^2 \cdot B_s \cdot \sin(\omega \cdot t)$$

... Eq. (6)

Substituting these two equations into Eq. (2) gives

$$-\omega^2 \cdot A_s \cdot \cos(\omega \cdot t) - \omega^2 \cdot B_s \cdot \sin(\omega \cdot t) + 2 \cdot \zeta \cdot \omega_n \cdot \left[-\omega \cdot A_s \cdot \sin(\omega \cdot t) + \omega \cdot B_s \cdot \cos(\omega \cdot t) \right] + \omega_n^2 \cdot \left[A_s \right.$$

$$\cdot\cos(\omega\cdot t)+B_s\cdot\sin(\omega\cdot t)\Big]=f_0\cdot\cos(\omega\cdot t)$$

... Eq. (7)

This can be further simplified to

$$\left(-\omega^2\cdot A_s+2\cdot\zeta\cdot\omega_n\cdot\omega\cdot B_s+\omega_n^{\,2}\cdot A_s-f_0\right)\cdot\cos(\omega\cdot t)+\left(-\omega^2\cdot B_s-2\cdot\zeta\cdot\omega_n\cdot\omega\cdot A_s+\omega_n^{\,2}\cdot B_s\right)\cdot\sin(\omega\cdot t)$$
$$=0$$

.. Eq. (8)

This equation must be valid for all values of t. When $t=0$, Eq. (9) simplifies to

$$\left(\omega_n^{\,2}-\omega^2\right)\cdot A_s+\left(2\cdot\zeta\cdot\omega_n\cdot\omega\right)\cdot B_s=f_0$$

... Eq. (9)

and when $t=\dfrac{\pi}{2\cdot\omega}$, Eq. (9) simplifies to

$$\left(-2\cdot\zeta\cdot\omega_n\cdot\omega\right)\cdot A_s+\left(\omega_n^{\,2}-\omega^2\right)\cdot B_s=0$$

... Eq. (10)

Solving these two equations, gives

$$A_s=\frac{\left(\omega_n^{\,2}-\omega^2\right)\cdot f_0}{\left(\omega_n^{\,2}-\omega^2\right)^2+\left(2\cdot\zeta\cdot\omega_n\cdot\omega\right)^2}$$

... Eq. (11)

and

$$B_s=\frac{\left(2\cdot\zeta\cdot\omega_n\cdot\omega\right)\cdot f_0}{\left(\omega_n^{\,2}-\omega^2\right)^2+\left(2\cdot\zeta\cdot\omega_n\cdot\omega\right)^2}$$

... Eq. (12)

Substituting Eqs. (11) and (12) back into Eq. (4) and rewriting it in the form of Eq. (3), gives

$$x_p(t) = \frac{f_0}{\sqrt{\left(\omega_n^2 - \omega^2\right)^2 + \left(2\cdot\zeta\cdot\omega_n\cdot\omega\right)^2}} \cdot \cos\left(\omega\cdot t - \tan^{-1}\left(\frac{2\cdot\zeta\cdot\omega_n\cdot\omega}{\omega_n^2 - \omega^2}\right)\right)$$

... Eq. (13)

This is the equation for the particular solution. For the total solution, we need to add the homogeneous solution to this. For an underdamped system, we add Eq. (12) from the Free Response Part 2 module to get

$$x(t) = A\cdot e^{-\zeta\cdot\omega_n\cdot t} \cdot \sin\left(\omega_d\cdot t + \phi\right) + X\cdot\cos(\omega\cdot t - \theta)$$

... Eq. (14)

where, using the initial conditions $x_0 = x(0)$ and $v_0 = \dot{x}(0)$, the constant are given by

$$\phi = \tan^{-1}\frac{\omega_d\cdot\left(x_0 - X\cdot\cos(\theta)\right)}{v_0 + \left(x_0 - X\cdot\cos(\theta)\right)\cdot\zeta\cdot\omega_n - \omega\cdot X\cdot\sin(\theta)},\ A = \frac{x_0 - X\cdot\cos(\theta)}{\sin(\phi)},\ \theta$$

$$= \tan^{-1}\frac{\left(2\cdot\zeta\cdot\omega_n\cdot\omega\right)}{\omega_n^2 - \omega^2}$$

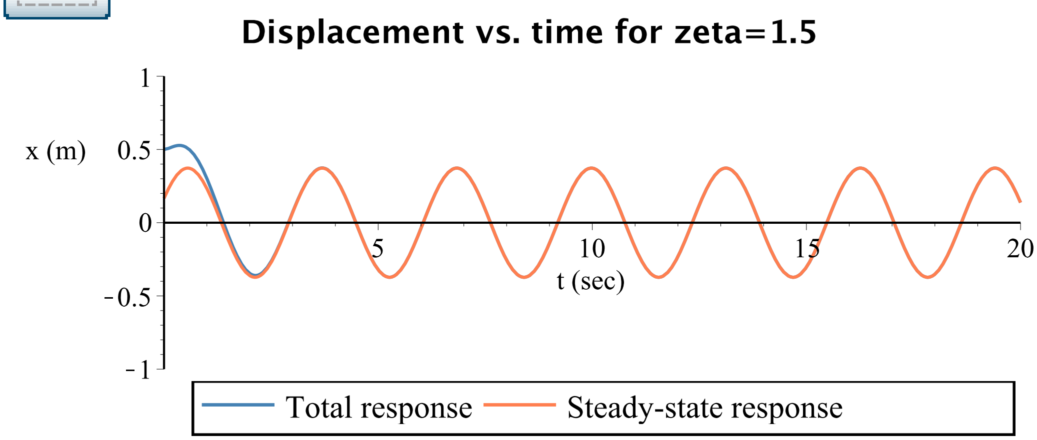
and $X = \dfrac{f_0}{\sqrt{\left(\omega_n^2 - \omega^2\right)^2 + \left(2\cdot\zeta\cdot\omega_n\cdot\omega\right)^2}}$

... Eqs. (15) to (18)

Eq. (14) is the total solution for an underdamped system. The total solutions for a critically damped or overdamped system can be found using the same method of adding the homogeneous solution to the particular solution and using the initial conditions to find the constants.

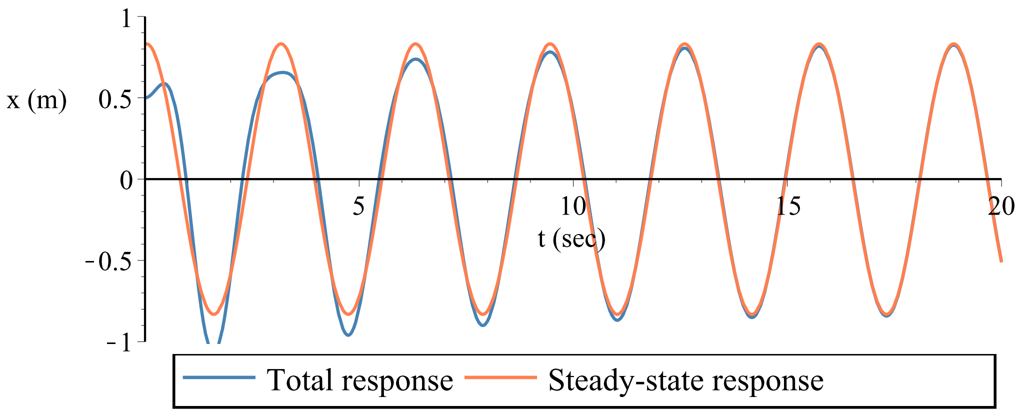
From Eq. (14) you can see that for relatively large values of t , the homogeneous part of the total solution approaches zero (this also applies to critically damped and overdamped systems because the homogeneous solutions for these two types of systems also decay with time). This means that as time increases, the total solution approaches the particular solution, which is also why the particular solution is also called the steady state response and the homogeneous solution is called the transient response. The following plots show a comparison of the total response and the steady-state response for a spring-mass system with different damping coefficients (the initial conditions and other parameters are kept constant: $k = 16\ \frac{N}{m}$, $m = 1\ kg$, $F_0 = 10\ N$, $\omega = 2\ \frac{rad}{s}$, $x_0 = 0.5\ m$, $v_0 = 0\ \frac{m}{s}$).

Comparison for zeta=1.5



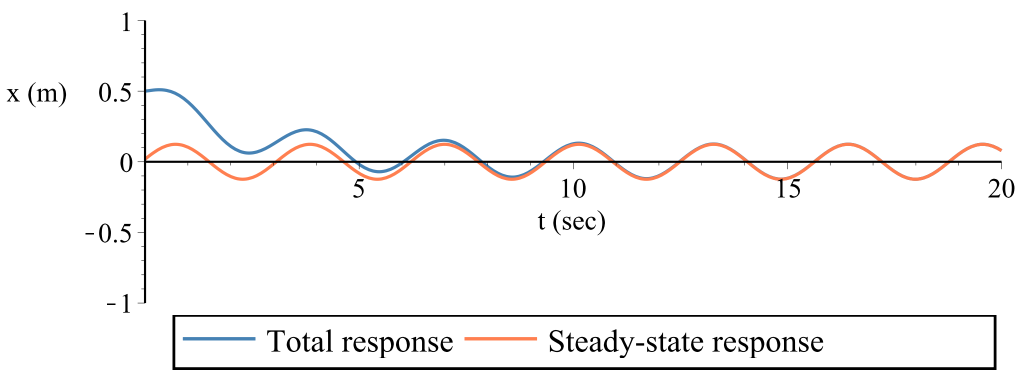
Comparison for zeta=0.05

Displacement vs. time for zeta=0.05



Comparison for zeta=5

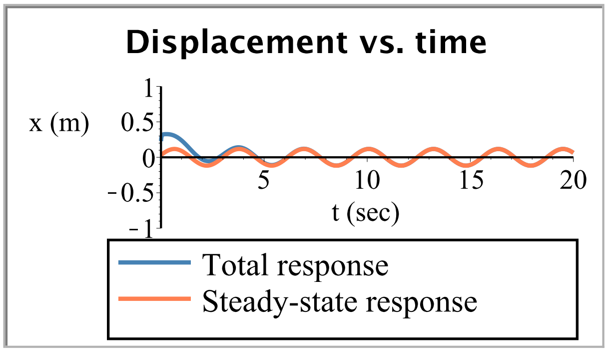
Displacement vs. time for zeta=5

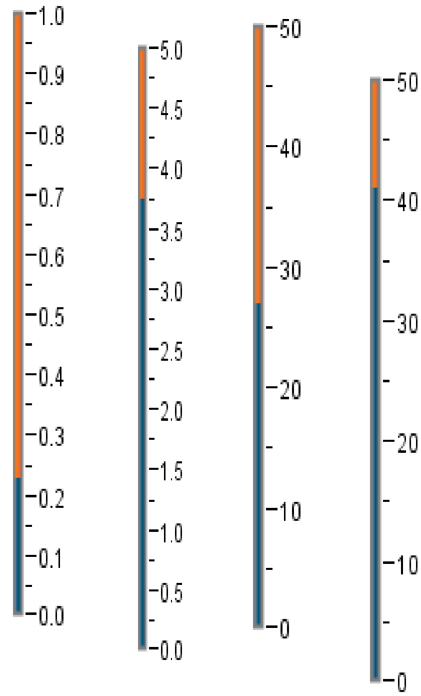


For the following plot, the damping coefficient, the spring constant and the initial conditions can be adjusted using the gauges to see the effect on the total response. The rest of the parameters of the system are the same as for the above plots (

$m = 1\ kg, F_0 = 10\ N, \omega = 2\ \frac{rad}{s}$).

$x_0\ (m)$ $v_0\ (m / s)$ $k\ (1 / mN)$ $c\ (1 / m(N \cdot s\backslash$





))

As can be seen from the plots, the total response and the steady-state response merge after some time. In many cases, depending on the system and the purpose of the analysis, the transient response may die out very quickly and may be ignored. These plots also show that the amplitude of the steady-state vibrations depend on the damping coefficient.

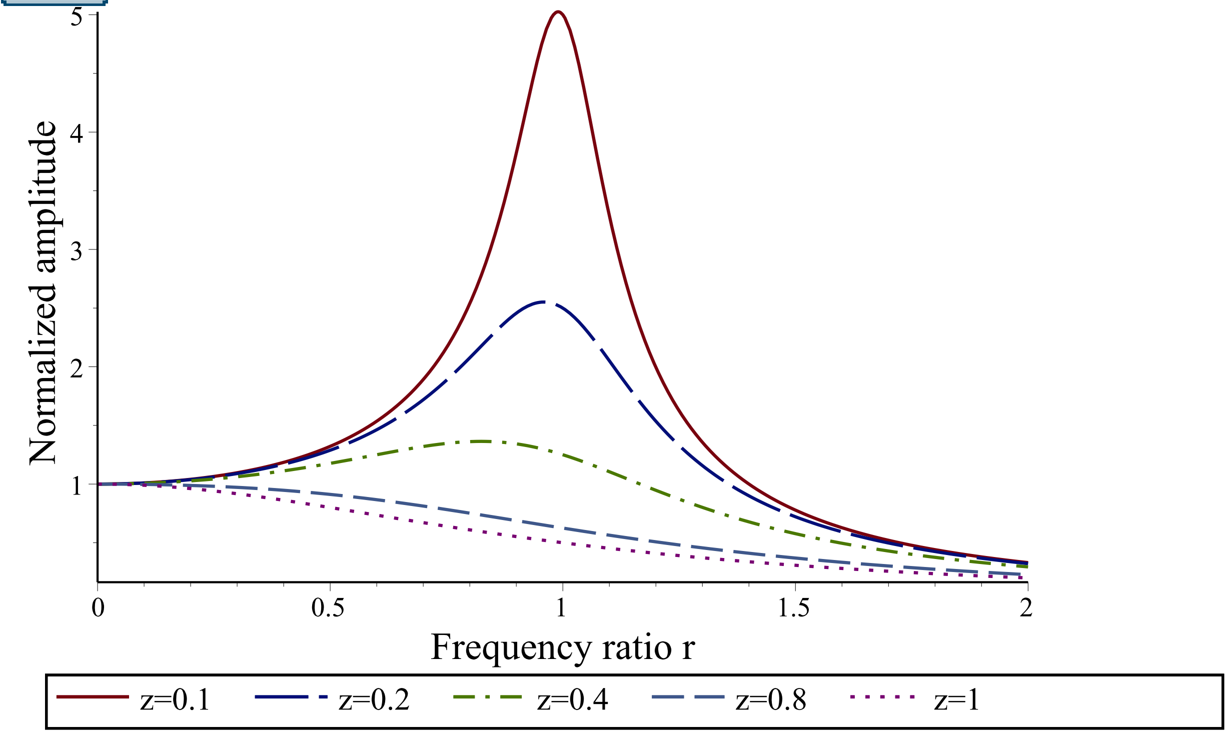
From Eq. (18), it is clear that the amplitude of the steady state vibrations also depends on the driving frequency. By defining another ratio, $r = \frac{\omega}{\omega_n}$, called the frequency ratio, Eq. (18) can be rewritten as the following equation for the normalized amplitude of the steady-state response.

$$\frac{X \cdot \omega_n^{\,2}}{f_0} = \frac{X \cdot k}{F_0} = \frac{1}{\sqrt{(1 - r^2)^2 + (2 \cdot \zeta \cdot r)^2}}$$

... Eq. (19)

This equation is plotted vs. the frequency ratio for different values of ζ:

Normalized amplitude vs. frequency ratio



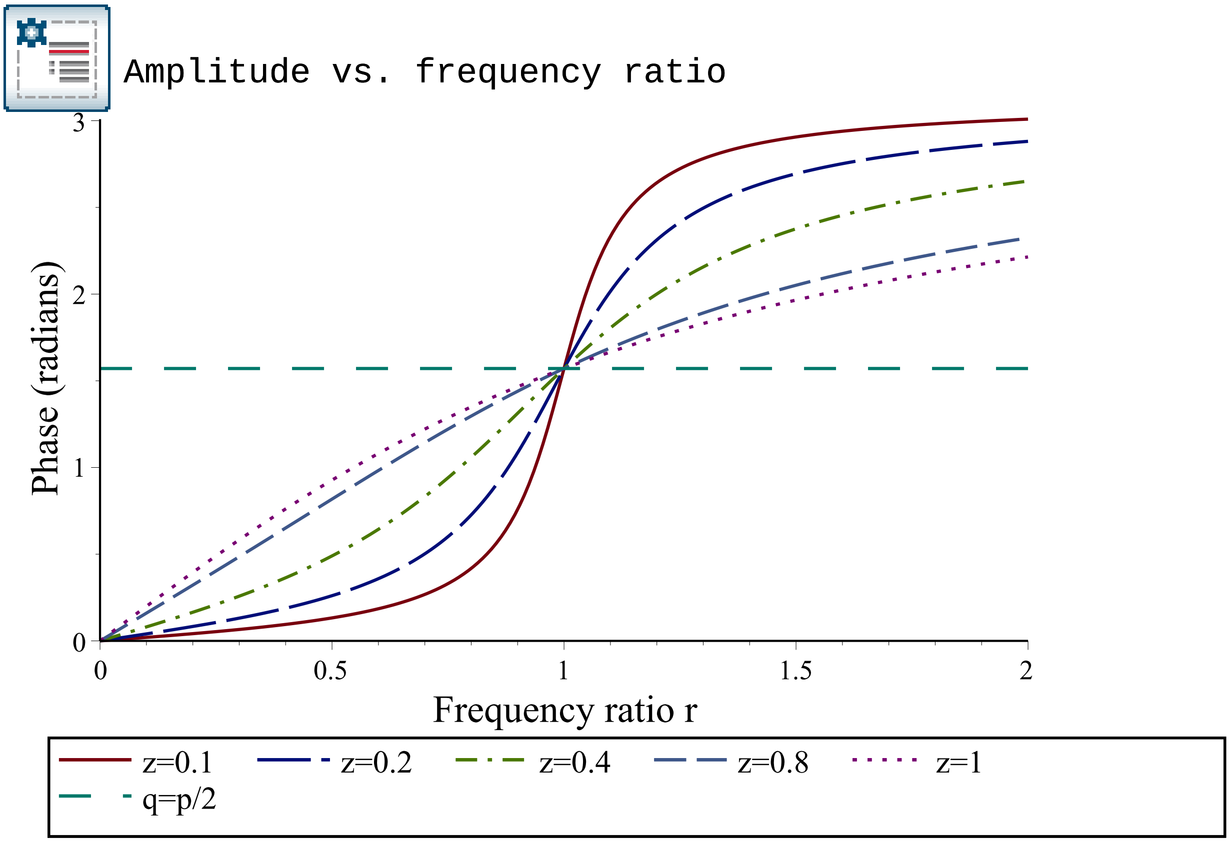
From this plot, it can be seen that for systems with low damping coefficients, the amplitude is maximum close to $r = 1$ (when the driving frequency is equal to the natural frequency). Also, it is important to note that, as the damping increases, the frequency for which the maximum amplitude is obtained shifts away from the natural frequency until there is no peak. The maximum amplitude is obtained when $r = \sqrt{1 - 2 \cdot \zeta^2}$ if $0 \le \zeta < \frac{1}{\sqrt{2}}$ and there is no peak if $\zeta > \frac{1}{\sqrt{2}}$. This information can be very important from a design point of view.

Similar to Eq. (19) the following equation shows the phase of the steady-state response as a function of the frequency ratio r.

$$\theta = \tan^{-1}\left(\frac{2 \cdot \zeta \cdot r}{1 - r^2}\right)$$

... Eq. (20)

The following plot shows the phase vs. the frequency ratio.



This plot shows that as the driving frequency increases, the phase difference between the steady-state response and the driving force increases from 0 to Pi and is equal to Pi/2 (90°) at the natural frequency of the system.

Examples with MapleSim

Example 1: Spring-mass oscillator

Compute the steady-state response of a spring-mass system with damping for the values given below and plot the response for 10 seconds.

Table 1: Example 1 parameter values

Parameter	Value
k	1000 N/m
c	100 N·s/m
m	10 kg
F_0	100 N
ω	8.16 rad/s
x_0	0.01 m
v_0	0.01 m/s

Analytical Solution

restart :

Data:

$k := 1000 :$ [N/m]

$m := 10 :$ [kg]

$c := 100 :$ [N·s/m]

$F_0 := 100 :$ [N]

$\omega := 8.16 :$ [rad/s]

$x_0 := 0.01 :$ [m]

v_0 [m/s]

$:= 0.\backslash$
$01 :$

Solution:

The natural frequency is

$$\omega_n := \sqrt{\frac{k}{m}} = 10$$

and the damping ratio is

$$\zeta_d := \frac{c}{2\cdot\sqrt{k\cdot m}} = \frac{1}{2}$$

Using Eq. (13) the steady-state response is given by

$$x := X\cdot\cos(\omega\cdot t - \theta) :$$

where

$$f_0 := \frac{F_0}{m} :$$

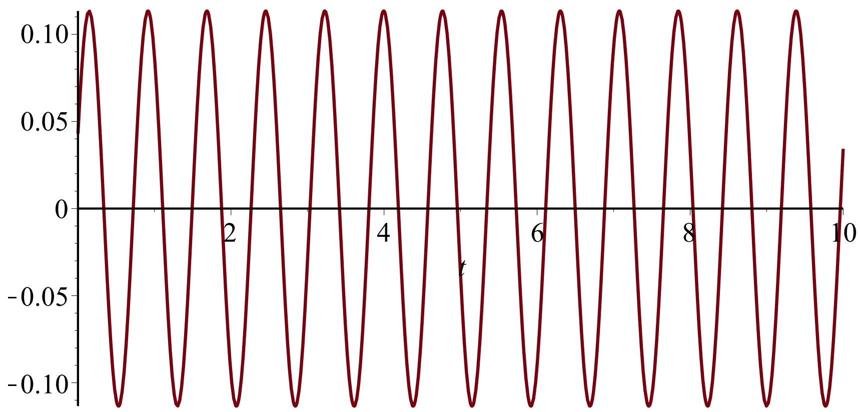
$$X := \frac{f_0}{\sqrt{\left(\omega_n^{\ 2} - \omega^2\right)^2 + \left(2\cdot\zeta_d\cdot\omega_n\cdot\omega\right)^2}} :$$

$$\theta := \tan^{-1}\left(\frac{2\cdot\zeta_d\cdot\omega_n\cdot\omega}{\omega_n^{\ 2} - \omega^2}\right) :$$

The expression for the steady-state response, x, is

$$0.1134089965 \cos(8.16\, t - 1.182135612) \qquad \textbf{(2.1.1.1)}$$

$plot(x, t = 0 ..10)$

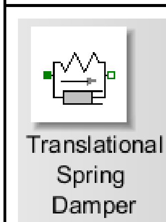
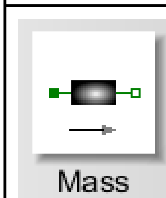


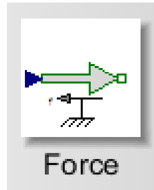
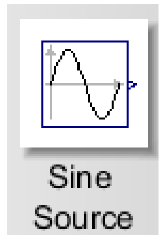
MapleSim Simulation

Step1: Insert Components

Drag the following components into the workspace:

Table 2: Components and locations

Component	Location
	1-D Mechanical > Translational > Common
	1-D Mechanical > Translational > Common
	1-D Mechanical > Translational > Common

	1-D Mechanical > Translational > Common
	Signal Blocks > Common

Step 2: Connect the components

Connect the components as shown in the following diagram.

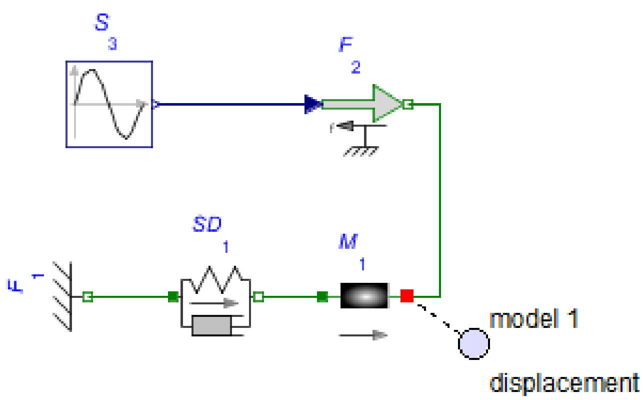


Fig. 2: Component diagram

Step 3: Set parameters and initial conditions

1. Click the **Translational Spring Damper** component, enter **1000** *N/m* for the spring constant (***c***), and enter **100** *N·s/m* for the damping constant (***d***).
2. Click the **Mass** component and enter **10** *kg* for the mass (***m***), **0.01** *m/s* for the initial velocity (v_0) and **0.01 m** for the initial position (s_0). Select the check marks that enforce these initial condition.

3. Click the **Sine Source** component and enter **100** for the amplitude, **8.16** *rad/s* for the frequency ***(f)*** and **Pi/2** for the phase **(ϕ)** [since we are assuming that the excitation is a cosine function].

Step 4: Run the Simulation

1. Attach a **Probe** to the **Mass** component as shown in Fig. 2. Click this **Probe** and select **Length** in the **Inspector** tab. This shows the position of the mass as a function of time.
2. Click **Run Simulation** ( ).

Example 2: Spring-Pendulum

Problem Statement: A component of a machine is modeled as a pendulum connected to a spring (as shown in Fig. 3). This component is driven by a motor that applies a sinusoidal moment $M(t) = 10\cos(4\cdot\mathrm{Pi}\cdot t)$Nm about the axis of rotation. Derive the equation of motion and find the natural frequency of the system. The mass of the pendulum is 2kg, the length of the pendulum is 0.5m, the stiffness of the spring is 20 N/m and the damping coefficient is 20 N·s/m. Assume that the rod of the pendulum has no mass. The initial angular displacement is 0.175 rad (approx. 10°) measured from the vertical and there is no initial angular velocity.

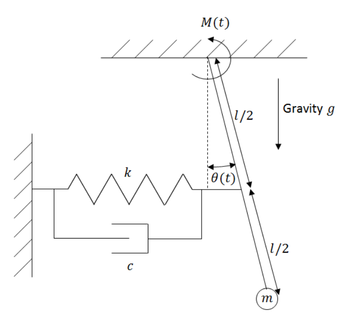


Fig. 3: Spring-pendulum example

a) Plot the total response and find the maximum angular displacement of the pendulum.
b) Find the angular frequency of the applied moment that would lead to steady state vibrations of the maximum amplitude.

Analytical Solution

Data:

restart :

$m := 2 :$	[kg]
$l := 0.5 :$	[m]
$k := 20 :$	[N/m]
$c := 20 :$	[N·s/m]
$g := 9.81 :$	[m/s^2]
$\theta_0 := 0.175 :$	[rad]
$\omega_0 := 0 :$	[rad/s]
$\omega := 4 \cdot \mathrm{Pi} :$	[rad/s]

Solution:

Part a) Total response

Using the small angle approximation, we will assume that the spring stretches and compresses in the horizontal direction only. Hence the force due to the spring can be written as

$$F_k := -k \cdot \left(\frac{l}{2}\right) \cdot \theta(t) :$$

and the force due to the damper can be written as

$$F_c := -c \cdot \left(\frac{l}{2}\right) \cdot \dot{\theta}(t)$$

Taking the sum of the moments of force about the pivot gives

$$J_0 \cdot \ddot{\theta}(t) = F_k \cdot \left(\frac{l}{2}\right) + F_c \cdot \left(\frac{l}{2}\right) - m \cdot g \cdot l \cdot \theta(t) + M(t) :$$

where J_0 is the moment of inertia of the pendulum. This equation can be rewritten as

$$m \cdot l^2 \cdot \ddot{\theta}(t) + \left(c \cdot \left(\frac{l}{2}\right)^2\right) \cdot \dot{\theta}(t) + \left(k \cdot \left(\frac{l}{2}\right)^2 + m \cdot g \cdot l\right) \cdot \theta(t) = 10 \cos(4 \cdot \mathrm{Pi} \cdot t) :$$

or

$$\ddot{\theta}(t) + \frac{c}{4 \cdot m} \cdot \dot{\theta}(t) + \frac{\left(\frac{k}{4} + \frac{m \cdot g}{l}\right)}{m} \cdot \theta(t) = \frac{10}{m \cdot l^2} \cos(4 \cdot \mathrm{Pi} \cdot t):$$

Comparing this equation to the form of Eq. (2) and using Eq. (14), the total response can be written as

$$\theta := A \cdot e^{-\zeta_d \cdot \omega_n \cdot t} \cdot \sin(\omega_d \cdot t + \phi) + X \cdot \cos(\omega \cdot t - \vartheta) :$$

where

$$\omega_n := \sqrt{\frac{\frac{k}{4} + \frac{m \cdot g}{l}}{m}} = 4.70$$

$$\zeta_d := \frac{\frac{c}{4 \cdot m}}{2 \cdot \sqrt{m \cdot \left(\frac{k}{4} + \frac{m \cdot g}{l}\right)}} = 0.1328885174$$

$$\omega_d := \omega_n \cdot \sqrt{1 - \zeta_d^2} = 4.661477770$$

$$X := \frac{\frac{10}{m \cdot l^2}}{\sqrt{\left(\omega_n^2 - \omega^2\right)^2 + \left(2 \cdot \zeta_d \cdot \omega_n \cdot \omega\right)^2}} :$$

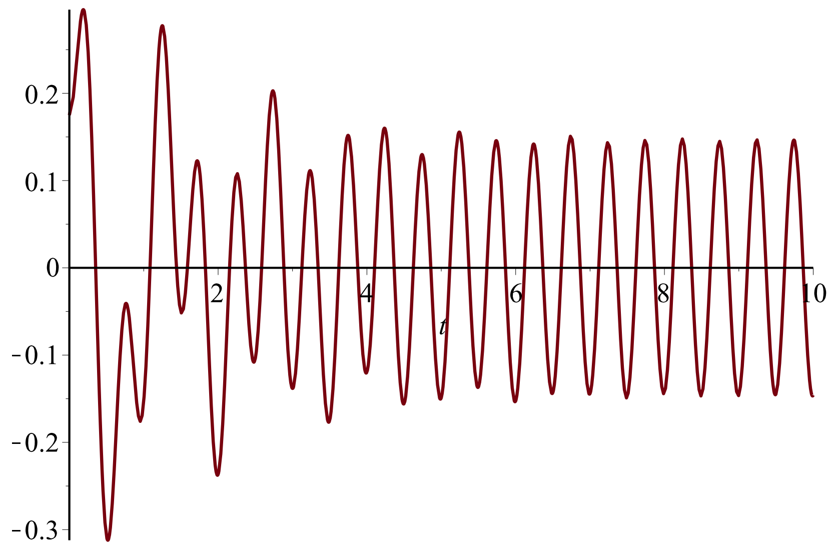
$$\vartheta := \tan^{-1}\left(\frac{2 \cdot \zeta_d \cdot \omega_n \cdot \omega}{\omega_n^2 - \omega^2}\right) + \mathrm{Pi} :$$

$$\phi := \tan^{-1}\left(\frac{\omega_d \cdot \left(\theta_0 - X \cdot \cos(\vartheta)\right)}{\omega_0 + \left(\theta_0 - X \cdot \cos(\vartheta)\right) \cdot \zeta_d \cdot \omega_n - \omega \cdot X \cdot \sin(\vartheta)}\right) :$$

$$A := \frac{\theta_0 - X \cdot \cos(\vartheta)}{\sin(\phi)}:$$

The following shows the plot of the total response for 10 seconds.

$$plot\left(A \cdot e^{-\zeta_d \cdot \omega_n \cdot t} \cdot \sin(\omega_d \cdot t + \phi) + X \cdot \cos(\omega \cdot t - \vartheta), t = 0..10\right)$$



From this plot, it can be concluded that the maximum displacement is approximately 0.29 rad at approximately 0.17 seconds.

Part b) Driving frequency for maximum amplitude steady-state vibrations

The amplitude of the steady-state oscillations is given by

$$\frac{\frac{10}{m \cdot l^2}}{\sqrt{\left(\omega_n^2 - \omega_F^2\right)^2 + \left(2 \cdot \zeta_d \cdot \omega_n \cdot \omega_F\right)^2}}:$$

where ω_F is the driving frequency. To find the value of ω_F for which the amplitude will be maximum, this expression is differentiated with respect to ω_F and equated to zero.

$$0 = diff\left(\frac{\frac{10}{m \cdot l^2}}{\sqrt{\left(\omega_n^2 - \omega_F^2\right)^2 + \left(2 \cdot \zeta_d \cdot \omega_n \cdot \omega_F\right)^2}}, \omega_F\right)$$

$$0 = -\frac{10.00000000 \left(-4 \left(-\omega_F^2 + 22.12000000\right) \omega_F + 3.125000002 \omega_F\right)}{\left(\left(-\omega_F^2 + 22.12000000\right)^2 + 1.562500001 \omega_F^2\right)^{3/2}} \quad \textbf{(2.2.1.2.1)}$$

$$\xrightarrow{\text{solve for } \omega\_\_F}$$

$$\left[\left[\omega_F = 0.\right], \left[\omega_F = 4.619388488\right], \left[\omega_F = -4.619388488\right]\right] \quad \textbf{(2.2.1.2.2)}$$

Ignoring $\omega_F = 0$ and $\omega_F = -4.619388488$ (does not make physical sense), we can conclude that the amplitude of the steady state oscillations will be maximum when the driving frequency is approximately 4.62 rad/s which is very close to the natural frequency of 4.70 rad/s.

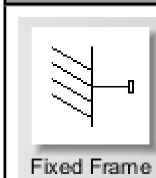
MapleSim Simulation

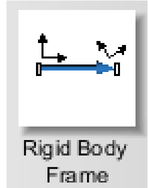
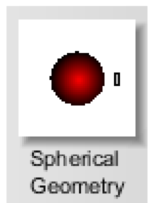
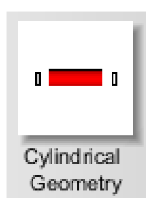
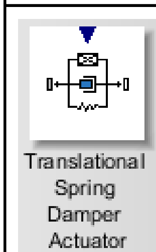
Constructing the model

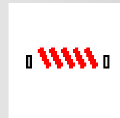
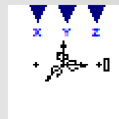
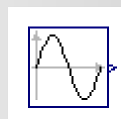
Step1: Insert Components

Drag the following components into the workspace:

Table 3: Components and locations

Component	Location
 (2 required)	Multibody > Bodies and Frames
	Multibody > Joints and Motions

 (2 required)	Multibody > Bodies and Frames
	Multibody > Bodies and Frames
	Multibody > Visualization
	Multibody > Visualization
	Multibody > Visualization
	Multibody > Forces and Moments
	Multibody >

 Spring Geometry	Visualization
 Applied World Moment	Multibody > Forces and Moments
 Sine Source	Signal Blocks > Common > Sine Source

Step 2: Connect the components

Connect the components as shown in the following diagram (the dashed boxes are not part of the model, they have been drawn on top to help make it clear what the different components are for).

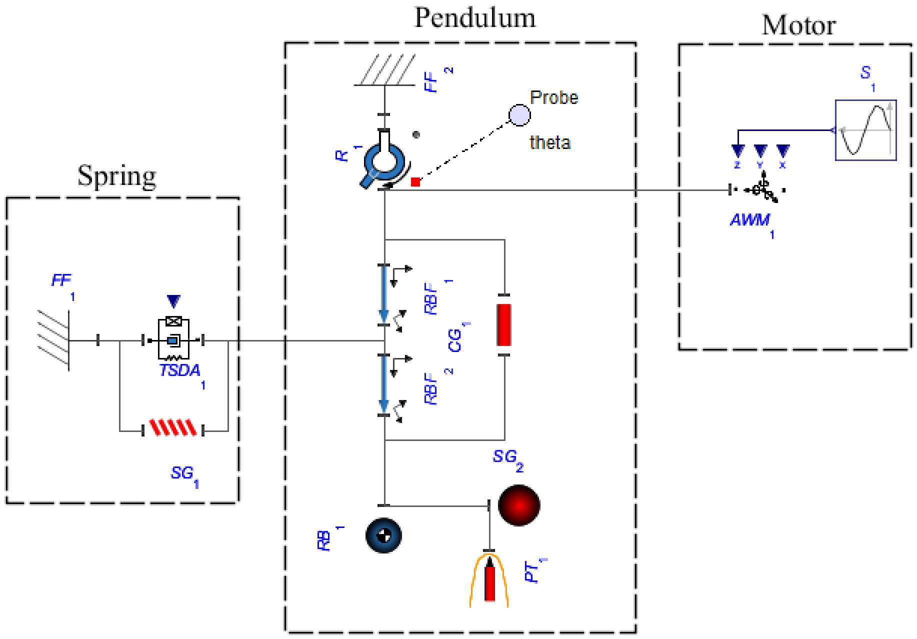


Fig. 4: Component diagram

Step 3: Set up the Pendulum

1. Click the **Revolute** component and enter **0.175** *rad* for the initial angle **(θ_0)** and select **Strictly Enforce** in the drop down menu for the initial conditions **($IC_{\theta,\omega}$)**. The axis of rotation ($\widehat{e}_1$) should be left as the default axis **[0,0,1]**.
2. Enter **[0,-0.25,0]** for the x,y,z offset **($\bar{r}_{XYZ}$)** of both the **Rigid Body Frames**.
3. Enter **2** *kg* for the mass **(m)** of the **Rigid Body Frame**.

Step 4: Set up the Spring

1. Click the **Fixed Frame** component connected to the **TSDA** (**FF_1** in the diagram) and enter **[-0.25,-0.25,0]** for the x,y,z offset **($\bar{r}_{XYZ}$)**.
2. Click the **TSDA** component, enter **20** *N/m* for the spring constant **(K_{spring})** and enter **20** *N·s/m* for the damping constant **(K_{damper})**. Also, enter **0.25** *m* for the unstretched length **(l_0)** to correspond to the location of the **Fixed Frame**.

Step 5: Set up the external sinusoidal moment (Motor)

1. Click the **Sine Source** component and enter **10** for the amplitude, **4*Pi** *rad/s* for the frequency **(*f*)** and **Pi/2** for the phase **(ϕ)** [the external force is a cosine function].
2. Connect the output of the **Sine Source** component to the **z** input of the **Applied World Moment** component.

Step 6: Set up the visualization (Inserting the **Visualization** components is optional)

1. Click the **Cylindrical Geometry** component and enter a value around **0.01** *m* for the radius.
2. Click the **Spherical Geometry** component and enter a value around **0.05** *m* for the radius.
3. Click the **Spring Geometry** component, enter a number around **10** for the number of windings, enter a value around **0.02** *m* for **radius1** and enter a value around **0.005** *m* for **radius2**.

Step 7: Run the Simulation

1. Click the **Probe** attached to the **Revolute** joint and select **Angle** to obtain a plot of the angular position vs. time.
2. Click **Run Simulation** ( ).

Since the analytical solution makes a small angle approximation, there will be a very slight variation in the results of this simulation and the results of the analytical method. From the plot generated using the analytical approach, the maximum amplitude is found to be approximately 0.29 rad. And, from the plot generated using the simulation, the maximum amplitude is found to be approximately 0.28 rad.

The following image shows the plot obtained from the simulation that shows the angle with respect to time.

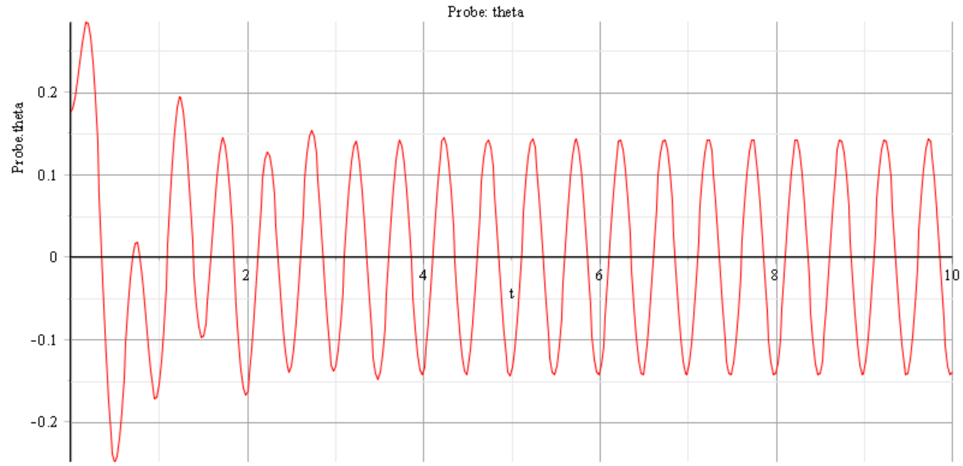
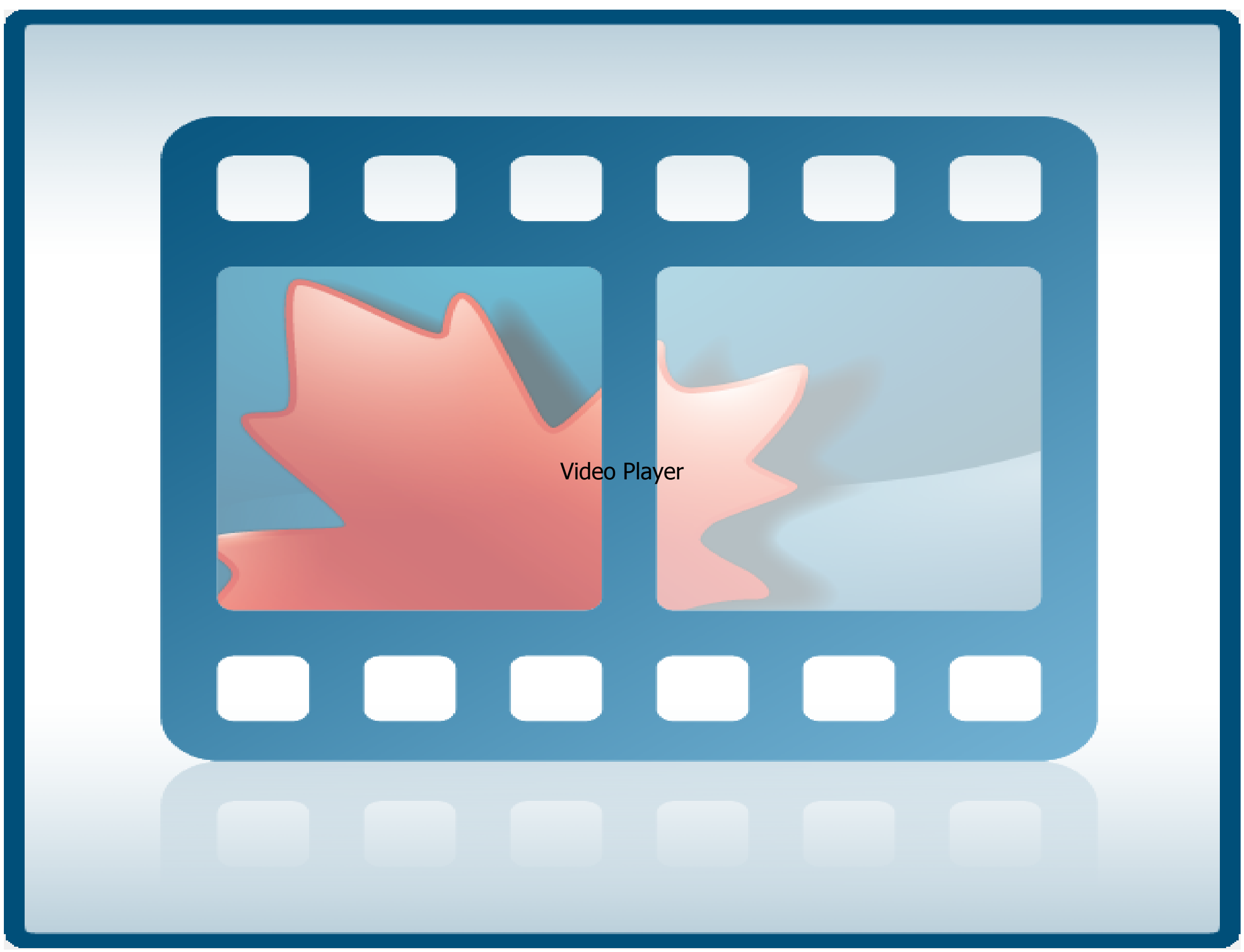


Fig. 5: Simulation results - theta (rad) vs. t (s)

The following video shows the visualization of the simulation



Video 1: Visualization of the spring-pendulum simulation with damping

Example 3: Wind turbine vibrations

Problem statement: The wind turbine described below has three rotor blades, one of which has a moment of inertia which is 1 percent less than the moment of inertia of the other two blades. Obtain a plot of the system response for a range of rotational speeds.

Description

The tower of a small wind turbine is a hollow steel cylinder of height 10 m with inner and outer diameters of 0.2 m and 0.15 m respectively. The density of the steel is 7800 Kg/m^3 and its modulus of elasticity is approximately 2 x10^{11} N/m^2. The mass of the nacelle and its contents (the generator and the drivetrain) is approximately 500 kg. The wind turbine has three rotor blades. Two of the blades blade have a mass of 10 Kg and a mass moment of inertia of approximately 20 kg/m^2. One of the blades has a moment of inertia which is 1% less than the other blades. This unbalance may be due to manufacturing defects, damage, wear, etc.

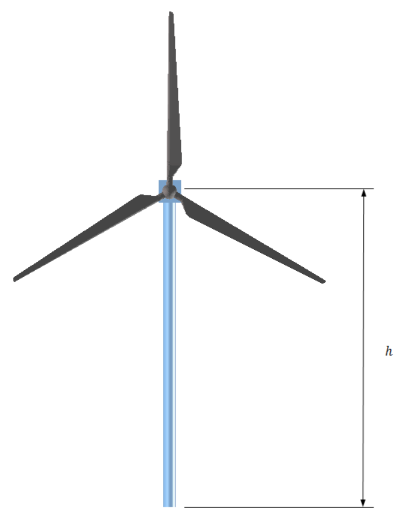


Fig. 6: Wind turbine

MapleSim Model

In this example, we will create a simplified lumped mass model of a wind turbine to study the vibrations due to unbalanced rotors.

The tower can be modeled as a cantilever beam with a tip mass subjected to a tip force. Based on the Euler-Bernoulli beam theory the deflection at the tip of a cantilever beam due to a tip force is

$$y_L = \frac{F \cdot L^3}{3 \cdot E \cdot I}$$

where F is the tip force, L is the length of the beam, E is the modulus of elasticity of the material and I is the area moment of inertia of the beam. This equation can be rewritten as

$$F = \frac{3 \cdot E \cdot I}{L^3} \cdot y_L$$

Therefore, the tower can be modeled as a spring with stiffness

$$k = \frac{3 \cdot E \cdot I}{L^3}$$

In this case,

$$I = \frac{\mathrm{Pi}}{64} \cdot \left(d_o^{\,4} - d_i^4\right)$$

where d_o is the outer diameter of the tower and d_i is the inner diameter.

To simplify the model, it will be assumed that the tower behaves as a rod hinged at the base. The mass moment of inertia of a rod about an axis passing through one end is

$$\frac{m_{rod} \cdot L^2}{3}$$

so the mass of the tower will be included in the model as a point mass of mass $\frac{m_{rod}}{3}$ located at the tip of the tower. The mass of the tower can be calculated using the following expression

$$\frac{\rho \cdot h \cdot \mathrm{Pi} \cdot \left(d_o^{\,2} - d_i^2\right)}{4}$$

where ρ is the density of the steel and h is the height of the tower. The mass of the nacelle and its contents (the generator and the drivetrain) is approximately 500 kg. This will also be modeled as a point mass at the tip of the tower. The wind turbine has three rotor blades. Each blade has a mass of 10 Kg and a mass moment of inertia of approximately 20 kg/m^2. Each blade can be modeled as a point mass of 10 kg located at a distance of $\sqrt{2}$ m from the axis of rotation. To study the effect of the unbalanced mass, the mass of one of the three blades will be set as $0.99 \cdot 10$ kg.

The following image shows the component diagram for the model.

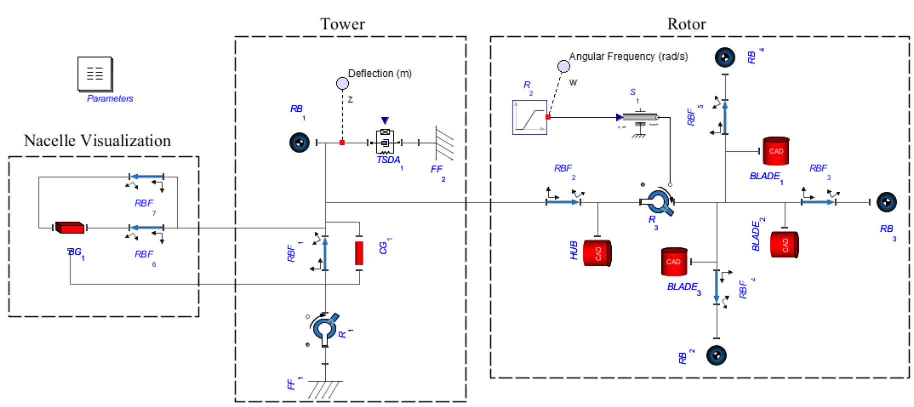


Fig. 7: Wind turbine simulation component diagram

As can be seen from this diagram, a **Ramp** component is used to apply a linearly increasing angular speed to the rotor. Also, CAD models for the rotor blades and the rotor hub have been attached for a more interesting visualization.

The following two images show the 3-D view of the simulation without and with visualization.

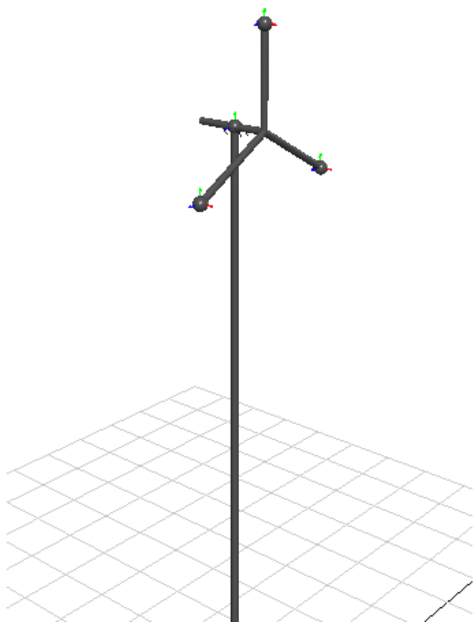


Fig. 8: 3-D view without CAD visualization

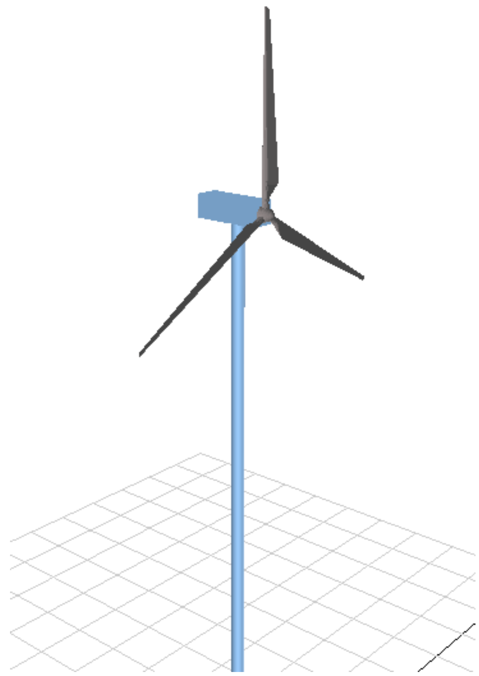


Fig. 9: 3-D view with CAD visualization

Simulation Results

The following two plots show the Angular Frequency of the rotation (in rad/s) vs. time (sec) and the deflection of the top of the turbine (m) vs. time (sec).

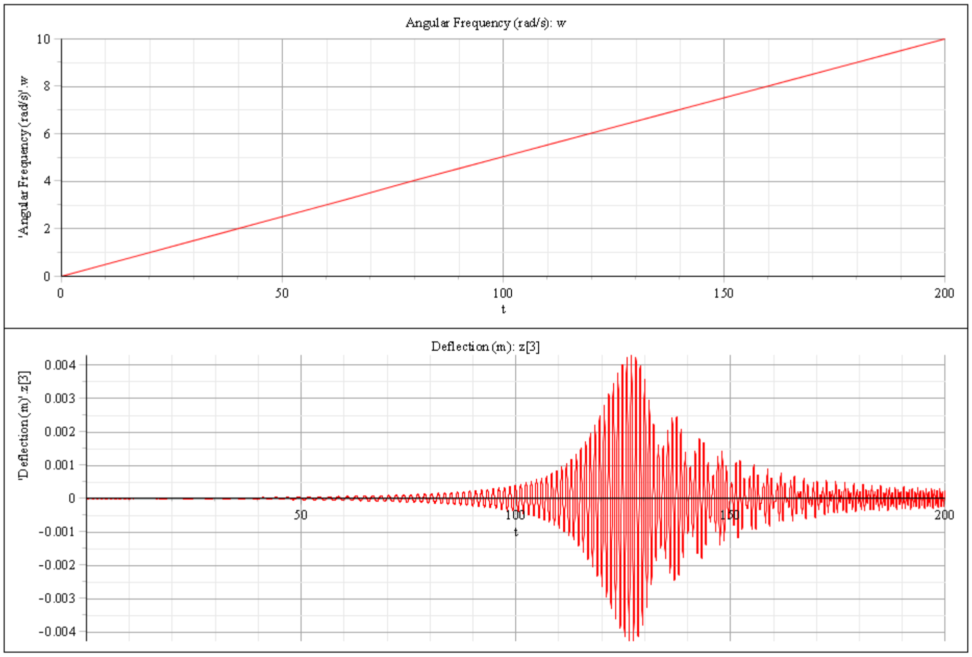


Fig. 10: Angular frequency (rad/s) and tip deflection (m) vs. time t (sec) plots for h=10 m.

These plots show that the greatest vibrations are obtained at around 6.5 rad/s. At this angular frequency the deflection of the top of the wind turbine is approximately 4 mm. This shows that a slight imbalance in the blades can result in relatively large deflections at certain speeds. Also, from a design point of view, this information is important because it shows what range of speeds should be avoided from a structural point of view and whether structural modifications are required.

If the height of the tower is reduced to 5 m and the range of speeds used is changed, the following results are obtained.

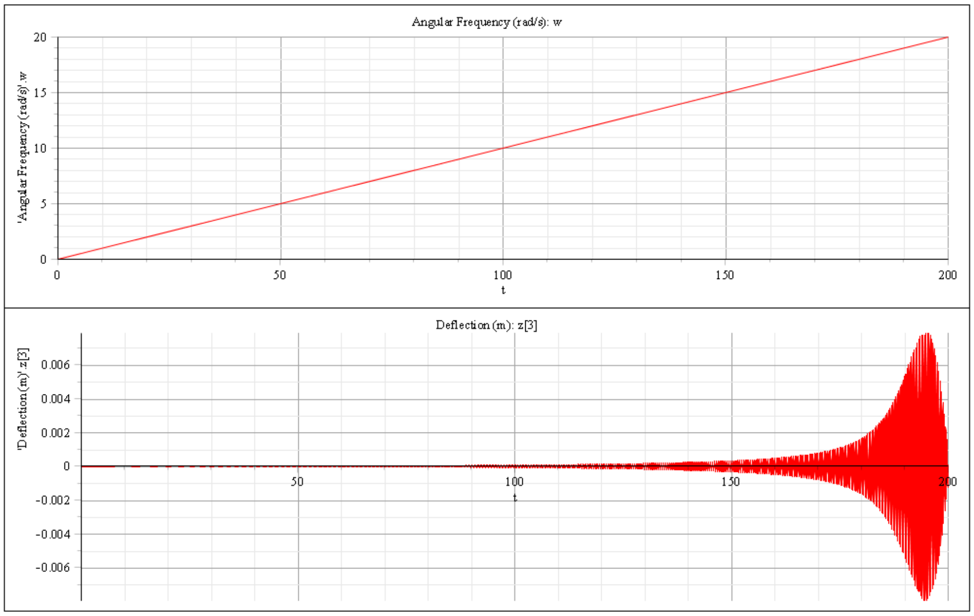
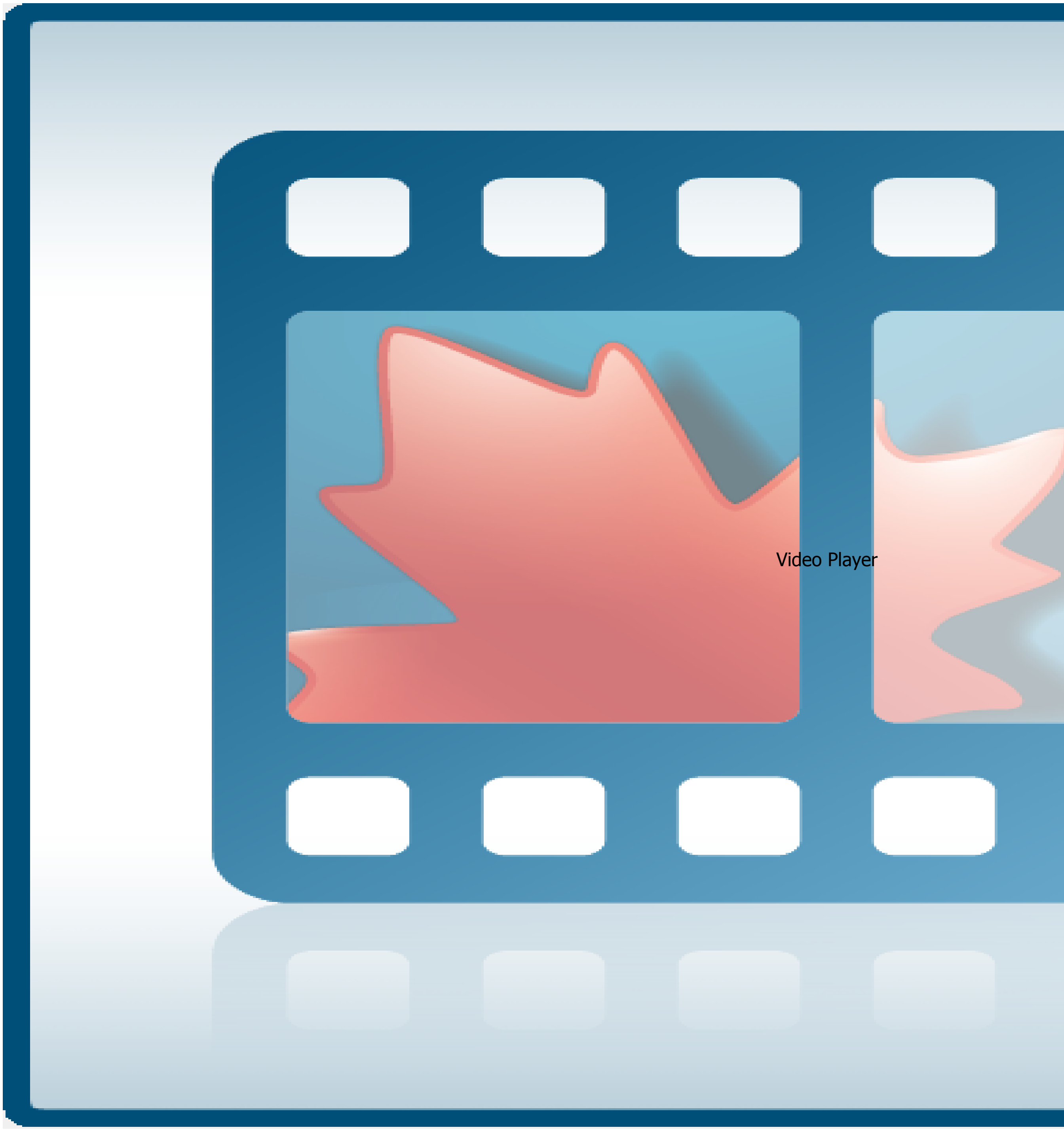


Fig. 11: Angular frequency (rad/s) and tip deflection (m) vs. time t (sec) plots for h=5 m.

In this case, the vibrations are maximum at a rotational speed around 19.5 rad/s.

The following video shows the 3-D visualization of the wind turbine.



Video 2: Wind turbine visualization

Reference:

D. J. Inman. "Engineering Vibration", 3rd Edition. Upper Saddle River, NJ, 2008, Pearson Education, Inc.